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On conditioned averages for intermittent turbulent flows

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A conditioning function is generally used to discriminate between the turbulent and non-turbulent zones in intermittent flows. Any fluid-mechanical property is conditioned by multiplying it by the intermittency function. A formalism is developed in which some integrals of the fluid-mechanical variables at the interface are important ingredients with a precise physical meaning. The present ideas may prove to be useful in turbulence dynamics and turbulent mixing of scalars.

The use of conditioned sampling techniques in turbulence is a relatively new procedure in experimental research (e.g. Libby 1975, 1976 and references therein) and an almost unexplored field for theoreticians. Recently Libby (1975, 1976) has developed some theoretical ideas, predicted some conditioned variables and compared the results with some available measurements. The crucial point in Libby's work is the postulate of a conservation equation for the intermittency function with an unknown source term. Some expressions for the latter have been proposed by Tutu (1976).

An indicator function, similar to the intermittency function, has been used in the description of the flow through 'non-swelling' porous media (e.g. Saffman 1971) and 'swelling' porous media (Dopazo & Corrsin 1977). There exists a clear analogy between the formal conditioned problems of intermittent flow and flow through porous media, with the obvious dynamic and boundary-condition differences. Saffman's (1971) formalism was correctly derived, made more accessible and extended to non-stationary nonlinear flow through deformable porous media by Dopazo & Corrsin (1977). This formalism is specialized here to the study of intermittent flows.

The equations governing the motion of a fluid in a turbulent/non-turbulent flow are

$$\partial u_i/\partial x_i = 0, \tag{1}$$

$$\frac{\partial u_i}{\partial t}+\frac{\partial u_i u_j}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\frac{1}{\rho}\frac{\partial \sigma_{ij}}{\partial x_j}, \tag{2}$$

with

$$\sigma_{ij} = \mu(\partial u_i/\partial x_j + \partial u_j/\partial x_i), \tag{3}$$

where $\mathbf{u}$ is the velocity, ρ the density, p the pressure, $\boldsymbol{\sigma}$ the viscous stress tensor and μ the viscosity. The intermittency function $I(\mathbf{x},t)$ is defined by

$$I(\mathbf{x},t) = \begin{cases} 1 & \text{if the point } (\mathbf{x},t) \text{ is in the turbulent region,} \\ 0 & \text{if the point } (\mathbf{x},t) \text{ is in the non-turbulent region.} \end{cases}$$

The intermittency is defined as the probability of $(\mathbf{x},t)$ being in the turbulent region, i.e.

$$\gamma(\mathbf{x},t) = \overline{I}(\mathbf{x},t), \tag{4}$$

where the overbar denotes the ensemble average. If the flow is statistically stationary the intermittency is also defined as the fraction of time that the point $\mathbf{x}$ is in the turbulent region, and time and ensemble averages are usually equated.

Let $Q(\mathbf{x}, t)$ be any fluid-mechanical property. It is easy to show that

$$\overline{\nabla(QI)} = \nabla(\overline{QI}) \tag{5}$$

using generalized functions or component notation. In vector notation, however, some precise arguments are necessary and are presented in Dopazo & Corrsin (1977). *Relation (5) contradicts Saffman's conclusion that 'the operations of differentiation* and ensemble averaging do not commute for discontinuous functions'. It can similarly be shown that

$$\overline{\partial(QI)/\partial t} = \partial(\overline{QI})/\partial t. \tag{6}$$

To condition the conservation equations the ensemble averages of $I\nabla Q$ and $I\,\partial Q/\partial t$ are needed. The mean value of the identity

$$I\nabla Q = \nabla(QI) - Q\nabla I \tag{7}$$

over a control volume $\mathscr{V}$ is thus taken. Note that ∇I is different from zero only at the interface; there ∇I has the absolute value of a Dirac delta-function and the direction of the normal $\mathbf{n}$ to the interface pointing towards the turbulent region. If one subsequently integrates the last term of (7) along the normal to the interface, lets $\mathscr{V}$ go to zero and takes the ensemble average of the remaining equality it is easy to show that

$$\overline{I\nabla Q} = \nabla(\overline{QI}) - \lim_{\mathscr{V}\to 0} \overline{\frac{1}{\mathscr{V}} \int_{S(\mathbf{x},t)} Q\mathbf{n}\, dS}, \tag{8}$$

where $S(\mathbf{x}, t) = 0$ is the equation of the turbulent/non-turbulent interface. An expression similar to (8) was presented by Saffman (1971) but with an incorrect interpretation of the terms; the notation used here is less ambiguous. Similarly,

$$I\frac{\partial Q}{\partial t} = \frac{\partial(QI)}{\partial t} - Q\frac{\partial I}{\partial t}. \tag{9}$$

Now $\partial I/\partial t = \delta(t - t_c)$, t_c being the time at which the interface crosses the point $\mathbf{x}$. Note that the last term in (9) arises from the motion of the interface and thus the height of the control volume is $\mathbf{u}^s dt.\mathbf{n}$, where $\mathbf{u}^s$ is the velocity of the interface. Integrating over t, letting $\mathscr{V}$ go to zero and ensemble averaging, one obtains

$$\overline{I\frac{\partial Q}{\partial t}} = \frac{\partial(\overline{QI})}{\partial t} + \lim_{\mathscr{V}\to 0} \overline{\frac{1}{\mathscr{V}} \cdot \int_{S(\mathbf{x},t)} Q\mathbf{u}^s . \mathbf{n}\, dS}. \tag{10}$$

The derivation of (8) and (10) appears in Dopazo & Corrsin (1977).

It is important to notice that the unconditioned fluctuation level Q' of any fluid-mechanical variable Q does not convey any direct information about the zone (turbulent or irrotational) fluctuations Q'_1 and Q'_0. The unconditioned mean $\bar{Q}$ enters the definition of Q', and hence Q' contains information from both the turbulent and the irrotational region. Let us use time averages as an illustration. Let T_T and T_N be the fractions of the total observation time T that the probe spends in the turbulent and irrotational fluid respectively. Similarly let Ω_T and Ω_N be the domains of integration for turbulent and irrotational zone averages. The topological union of Ω_T and Ω_N is equal to the total integration domain Ω. The total signal Q may then be decomposed into $Q_1 = IQ$ for $t \in \Omega_T$ and $Q_0 = (1-I)\,Q$ for $t \in \Omega_N$. The following averages are now defined:

$$\overline{IQ} = \lim_{T\to\infty} \frac{1}{T}\int_{\Omega} IQ\,dt = \lim_{T\to\infty}\frac{1}{T}\int_{\Omega_T} Q\,dt,$$

$$\overline{(1-I)\,Q} = \lim_{T\to\infty} \frac{1}{T}\int_{\Omega} (1-I)\,Q\,dt = \lim_{T\to\infty}\frac{1}{T}\int_{\Omega_N} Q\,dt.$$

Taking into account the fact that

$$T_T/T \to \gamma, \quad T_N/T \to 1-\gamma \quad \text{as} \quad T\to\infty,$$

the zone averages can be written as

$$\overline{Q}_1 = \lim_{T_T\to\infty}\frac{1}{T_T}\int_{\Omega_T} Q\,dt = \frac{\overline{IQ}}{\gamma},$$

$$\overline{Q}_0 = \lim_{T_N\to\infty}\frac{1}{T_N}\int_{\Omega_N} Q\,dt = \frac{\overline{(1-I)\,Q}}{1-\gamma}.$$

The real zone fluctuations will be defined by

$$Q_1' = Q_1 - \overline{Q}_1 = IQ - \overline{Q}_1 \quad \text{for} \quad t\in\Omega_T,$$

$$Q_0' = Q_0 - \overline{Q}_0 = (1-I)\,Q - \overline{Q}_0 \quad \text{for} \quad t\in\Omega_N.$$

Therefore

$$\overline{Q_1'} = \lim_{T_T\to\infty}\frac{1}{T_T}\int_{\Omega_T}(IQ - \overline{Q_1})\,dt = 0$$

and similarly $\overline{Q_0'} = 0$.

For the products of the fluctuations of two random variables P and Q we have

$$P_1'Q_1' = IPQ - \overline{P}_1 IQ - \overline{Q}_1 IP + \overline{P}_1\overline{Q}_1 \quad \text{for} \quad t\in\Omega_T.$$

Zone averaging yields

$$\overline{P_1'Q_1'} = \lim_{T_T\to\infty}\frac{1}{T_T}\int_{\Omega_T} P_1'Q_1'\,dt = \frac{\overline{IPQ}}{\gamma} - \overline{P}_1\overline{Q}_1.$$

This decomposition is simpler than Libby's (1975, formula (9)) and does not mix conditioned and unconditioned averages in the same definition, thanks to the use of real zone fluctuations.

Use of (8) and (10) with $Q \equiv 1$ yields

$$\nabla\gamma = \lim_{\mathscr{V}\to\infty}\overline{\frac{1}{\mathscr{V}}\int_{S(\mathbf{x},t)}\mathbf{n}\,dS}, \tag{11}$$

$$\frac{\partial\gamma}{\partial t} = -\lim_{\mathscr{V}\to 0}\overline{\frac{1}{\mathscr{V}}\int_{S(\mathbf{x},t)}\mathbf{u}^s\,.\,\mathbf{n}\,dS}. \tag{12}$$

For statistically stationary flows (12) expresses the fact that positive and negative values of $\mathbf{u}^s\,.\,\mathbf{n}$ of the same magnitude are equally probable. The computation of (11) or (12) requires detailed knowledge of the interface dynamics, i.e. the equation

$S(\mathbf{x},t) = 0$, and thus the detailed geometry of the interface is involved. If $\mathbf{u}^s$ is decomposed into $\mathbf{u}$, the velocity of a fluid element, and $-V\mathbf{n}$, the speed of advance of an element of the surface relative to the fluid element at the same point (Phillips 1972), (12) may be written as

$$\frac{\partial \gamma}{\partial t} = -\overline{\lim_{\mathscr{V}\to 0} \frac{1}{\mathscr{V}} \int_S \mathbf{u}\,.\,\mathbf{n}\,dS} + \overline{\lim_{\mathscr{V}\to 0} \frac{1}{\mathscr{V}} \int_S V\,dS}, \tag{13}$$

where the last term can be considered as the entrainment per unit volume divided by the density ρ, i.e. the mass entrainment per unit mass.

The conditioned equations are obtained by multiplying (1) and (2) by I, and making use of (5), (6), (8) and (10)–(12). Equation (1) yields

$$\frac{\partial \gamma}{\partial t} + \frac{\partial(\gamma\overline{(u_1)_i})}{\partial x_i} = \overline{\lim_{\mathscr{V}\to 0} \frac{1}{\mathscr{V}} \int_S V\,dS}. \tag{14}$$

If the flow is statistically stationary (14) reduces to

$$\frac{\partial(\gamma\overline{(u_1)_i})}{\partial x_i} = \overline{\lim_{\mathscr{V}=0} \frac{1}{\mathscr{V}} \int_S V\,dS} \tag{15}$$

and $\overline{(u_1)}_j$, the conditioned mean velocity in the turbulent region, is non-solenoidal. Note that Libby's definition is different from the one above. Thus the ensemble average of $\dot{w}$ in Libby (1975) is identical to the right-hand side of (15), i.e. the entrainment per unit mass.

The conditioned momentum equation is

$$\frac{\partial(\gamma\overline{(u_1)_i})}{\partial t} + \frac{\partial(\overline{Iu_i u_j})}{\partial x_j} = \overline{\lim_{\mathscr{V}\to 0} \frac{1}{\mathscr{V}} \int_S \rho u_i\,V\,dS} - \frac{1}{\rho}\frac{\partial(\overline{Ip})}{\partial x_i} + \nu\nabla^2(\gamma\overline{(u_1)_i}) - F_i(\mathbf{x},t), \tag{16}$$

where

$$F_i(\mathbf{x},t) = \frac{1}{\rho}\overline{\lim_{\mathscr{V}\to 0} \frac{1}{\mathscr{V}} \int_S (-p\delta_{ij} + \sigma_{ij})\,n_j\,dS} + \nu\frac{\partial}{\partial x_j}\overline{\lim_{\mathscr{V}\to 0} \frac{1}{\mathscr{V}} \int_S (u_i n_j + u_j n_i)\,dS} \tag{17}$$

is the mean force per unit mass that the turbulent fluid exerts on the non-turbulent fluid. The first term on the right-hand side of (16) is the momentum flux through the interface, i.e. the ith component of the momentum entrained into the turbulent region from the potential-flow zone. The last term in (17) is an additional viscous force of the turbulent on the non-turbulent region. The physical meaning of the latter as well as a different mathematical way to recover it may be clearly seen in Landau & Lifshitz (1959) and Batchelor (1970) in connexion with the viscosity of suspensions. It was to be expected that the relative forces between turbulent and non-turbulent fluid should enter the conditioned dynamic equations. Multiplying (2) by I and taking the ensemble average is equivalent to isolating a control volume of the turbulent fluid from the rest of the flow; the interactive force between this control volume and the rest of the flow must hence be included in the dynamic equations for the resulting conditioned variables.

If the flow is statistically stationary (16) becomes

$$\frac{\partial(\overline{Iu_i u_j})}{\partial x_j} = \overline{\lim_{\mathscr{V}\to 0} \frac{1}{\mathscr{V}} \int_S \rho u_i\,V\,dS} - \frac{1}{\rho}\frac{\partial(\overline{Ip})}{\partial x_i} + \nu\nabla^2(\gamma\overline{(u_1)_i}) - F_i(\mathbf{x},t). \tag{18}$$

The conditioned energy tensor $\overline{Iu_i u_j}$ may be decomposed into

$$\overline{Iu_i u_j} = \gamma\overline{(u_1)_i}\,\overline{(u_1)_j} + \gamma\overline{(u_1')_i(u_1')_j} \tag{19}$$

and

$$\overline{Ip} = \gamma\overline{p_1}. \tag{20}$$

In (19) only variables evaluated in the turbulent fluid appear. This is different from Libby's decomposition, in which turbulent and non-turbulent averages are present.

Similar equations may be derived for the non-turbulent region by multiplying (1) and (2) and $1-I$, taking the ensemble average and using expressions similar to (5)–(12) with I replaced by $1-I$ (for the formal procedure see Dopazo & Corrsin 1977). F_i in (16) appears with the sign reversed in the equation for $(1-\gamma)\overline{(u_0)_i}$. The equations for $\overline{(u_1)_i}$ and $\overline{(u_0)_i}$ are thus coupled through this interaction term, i.e. through the interface dynamics. In the equations for the traditional unconditioned average velocity $\gamma\overline{(u_1)_i} + (1-\gamma)\overline{(u_0)_i}$, neither the entrainment nor the force F_i appears.

The conditioned kinetic energy equation can easily be obtained by following the above formal steps. An unknown variable that enters this equation is

$$W(\mathbf{x},t) = \overline{\lim_{\mathscr{V}\to 0}\frac{1}{\mathscr{V}}\int_S(-p\delta_{ij}+\sigma_{ij})\,n_j u_i\,dS}, \tag{21}$$

the mechanical work done by the turbulent fluid upon the non-turbulent fluid. The entrainment of turbulent kinetic energy also appears explicitly.

The equivalent to equation (14) in Libby's paper can easily be derived by averaging

$$(1-I)\,u_j(\partial u_i/\partial x_j - \partial u_j/\partial x_i) = 0. \tag{22}$$

After some manipulation one gets

$$\begin{aligned}\frac{\partial}{\partial x_j}(1-\gamma)\,\overline{(u_0)_i}\,\overline{(u_0)_j} - \frac{1}{2}\frac{\partial}{\partial x_i}(1-\gamma)\,\overline{(u_0)_j}\,\overline{(u_0)_j}\\ +\frac{\partial}{\partial x_j}(1-\gamma)\,\overline{(u_0')_i(u_0')_j} - \frac{1}{2}\frac{\partial}{\partial x_i}(1-\gamma)\,\overline{(u_0')_j(u_0')_j}\\ +\overline{\lim_{\mathscr{V}\to 0}\frac{1}{\mathscr{V}}\int_S(u_i u_j n_j - \tfrac{1}{2}u_j u_j n_i)\,dS} = 0.\end{aligned} \tag{23}$$

The last term in (23) is associated with momentum fluxes through the interface. In the limit $\gamma\to 0$ this term tends to zero and $(u_0)_i$ tends to a constant vector; (23) then becomes

$$\frac{\partial}{\partial x_j}\overline{(u_0')_i(u_0')_j} = \frac{1}{2}\frac{\partial}{\partial x_i}\overline{(u_0')_j(u_0')_j}, \tag{24}$$

which is the Corrsin–Kistler equation in our notation.

In order to solve (15) and (18) one still has to model the unknowns and close the conditioned Reynolds-stress terms in (19). An attempt to achieve the modelling is underway at the present time and these results will be the subject of a future paper.

Thanks to a formalism initiated by Saffman (1971), substantially extended and clarified by Dopazo & Corrsin (1977) in the context of porous media, and specialized here to turbulence, it is now possible to formulate conditioned intermittent turbulent flow equations in which physically meaningful unknown variables appear. This is an alternative to Libby's method, in which an equation for I with an unknown source term is postulated; postulating the equation for I is equivalent to specifying the interface dynamics, i.e. the equation $S(\mathbf{x}, t) = 0$, and it is not required in the above formulation. In the present formalism knowledge of the equation $S(\mathbf{x}, t) = 0$ is however necessary in order to compute the surface integrals entering the conditioned equations. In some respects the present formalism has advantages over Libby's: there is no need for an intermittency-function conservation equation, only physically meaningful unknowns appear in the conditioned equations and one expects to be able to model these terms more easily than those involving an unspecified $\dot{w}$ and derivatives of I. Once some progress has been achieved in understanding the geometry and dynamics of the interface the right-hand side of (11) will be directly predictable, and thus it will be possible for γ to be obtained. Some research along Phillips' (1972) lines seems to be needed.

The ideas contained in this brief note may prove to be of importance in the investigation of the interface dynamics, conditioned turbulence equations and turbulent mixing of reactive and passive scalars.

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